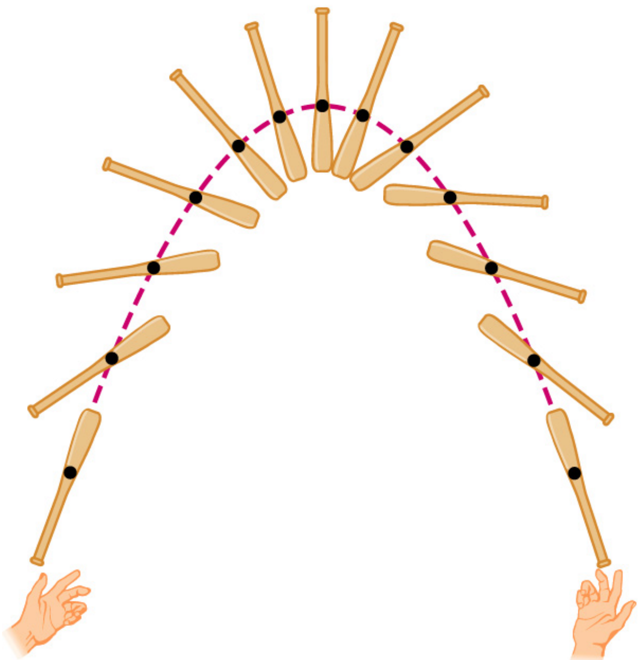
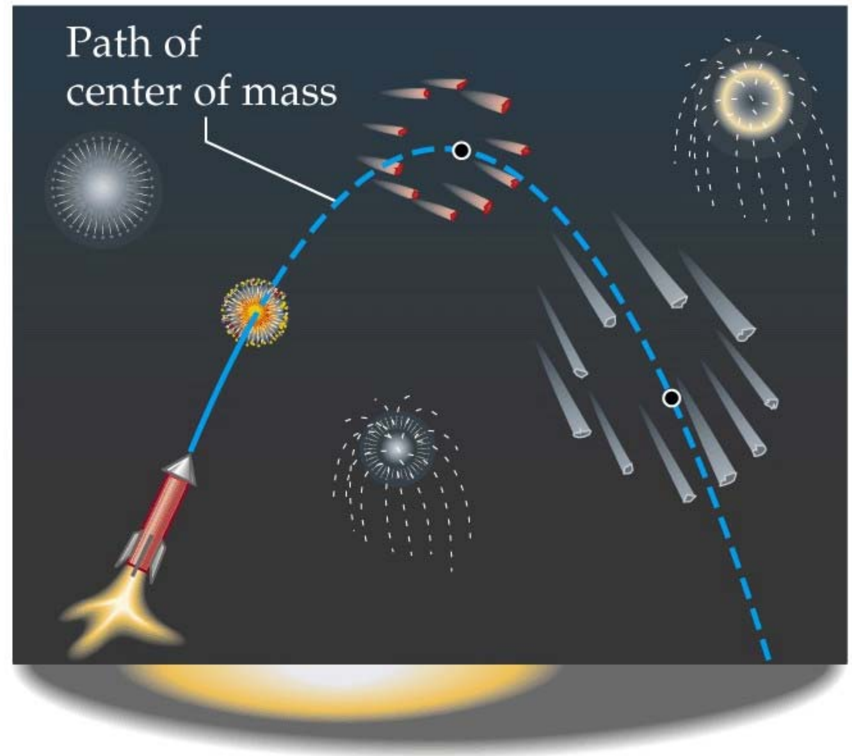
(a)

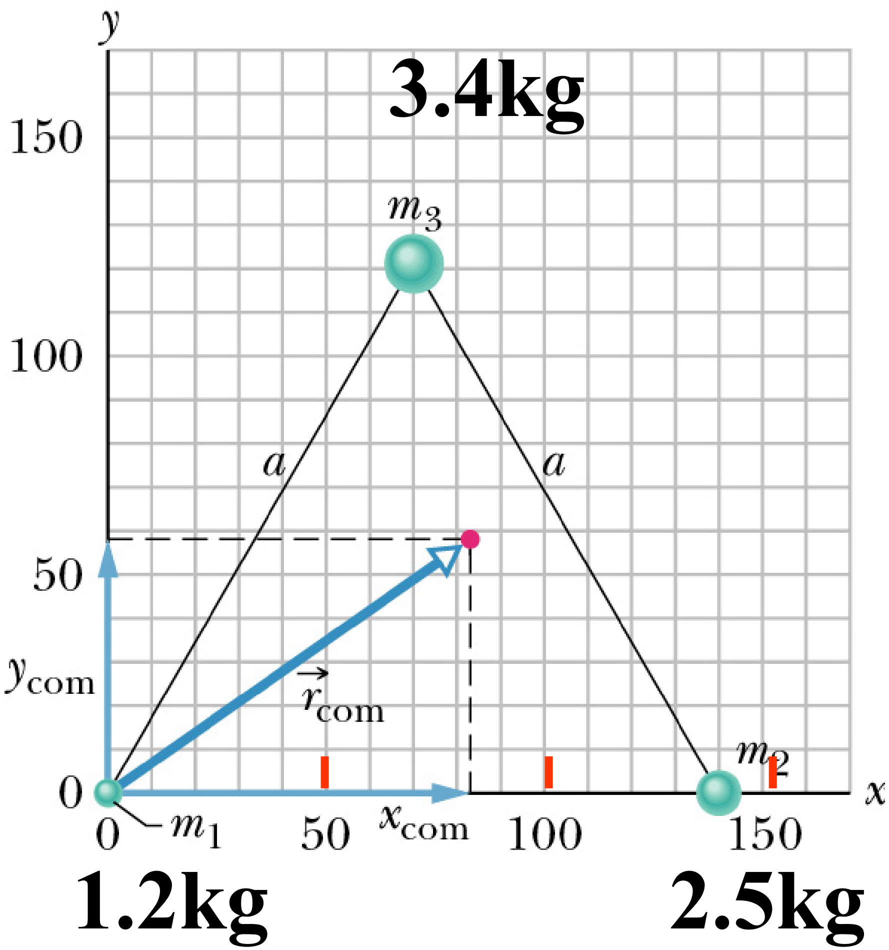


(b)



Środek masy

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + \ldots + m_n\vec{r}_n}{\vec{m}_1 + \ldots + m_n}$$



$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} =$$

$$\frac{1.2\cdot 0 + 2.5\cdot 140 + 3.4\cdot 70}{7.1} = 83\ cm$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = 58\ cm$$

Środek masy

- współrzędna środka masy x_{CM}:

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 \dots m_n x_n}{M} \quad gdzie \quad M = \sum_i m_i$$

- prędkość i pęd środka masy :

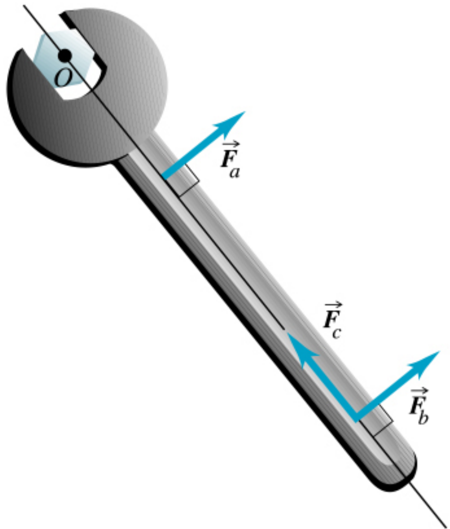
$$v_{CM} = \frac{\Delta x_{CM}}{\Delta t} = \frac{1}{M}\left(m_1 \frac{\Delta x_1}{\Delta t} + m_2 \frac{\Delta x_2}{\Delta t}\right)$$

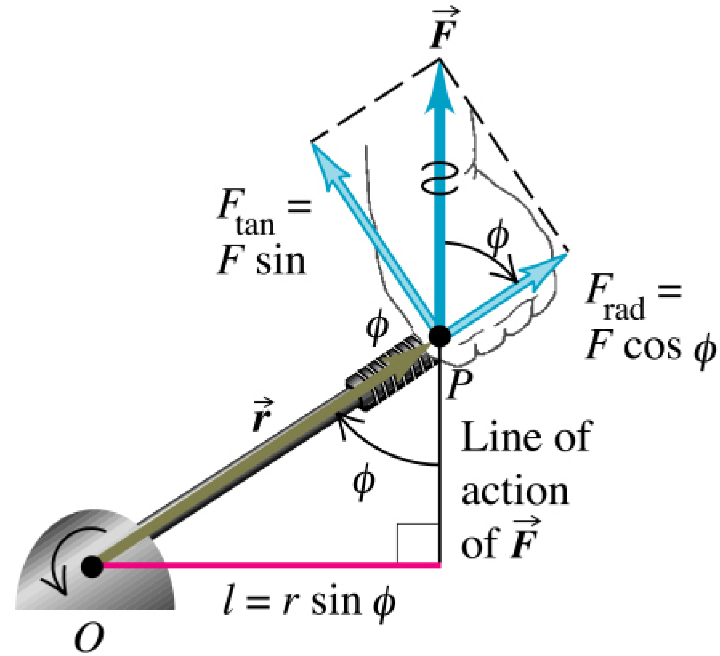
$$v_{CM} = \frac{1}{M}(m_1 v_1 + m_2 v_2) \Rightarrow p_{CM} = p_1 + p_2$$

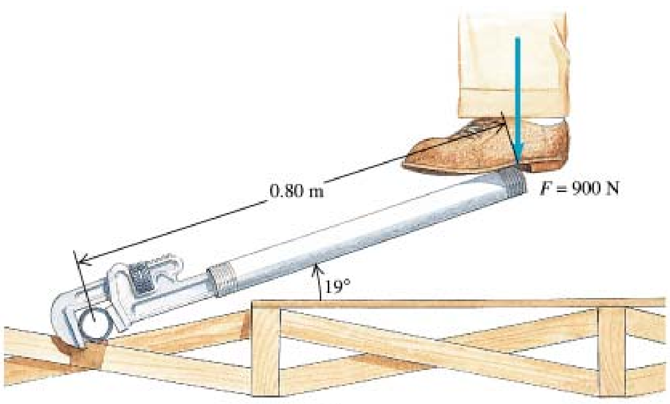
- przyspieszenie środka masy :

$$a_{CM} = \frac{\Delta v_{CM}}{\Delta t} = \frac{1}{M}\left(m_1 \frac{\Delta v_1}{\Delta t} + m_2 \frac{\Delta v_2}{\Delta t}\right) = \frac{1}{M}(m_1 a_1 + m_2 a_2)$$

$$a_{CM} = \frac{1}{M}(m_1 a_1 + m_2 a_2) = \frac{1}{M}(F_1 + F_2)$$



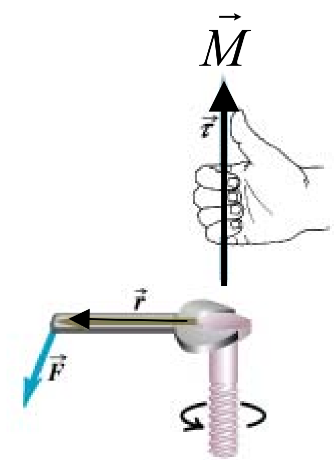




Moment siły

$$M = r\,F \sin \alpha$$

$$\vec{M} = \vec{r} \times \vec{F}$$

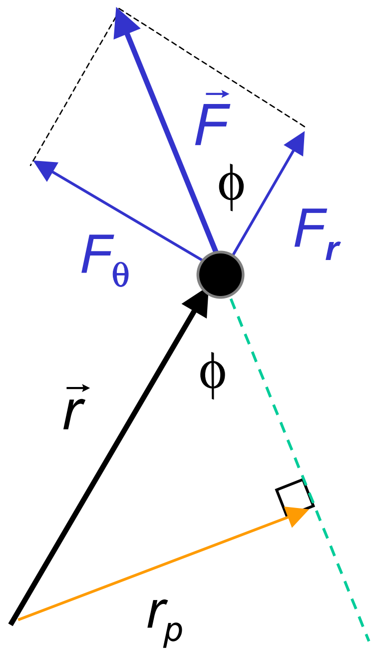


Moment siły

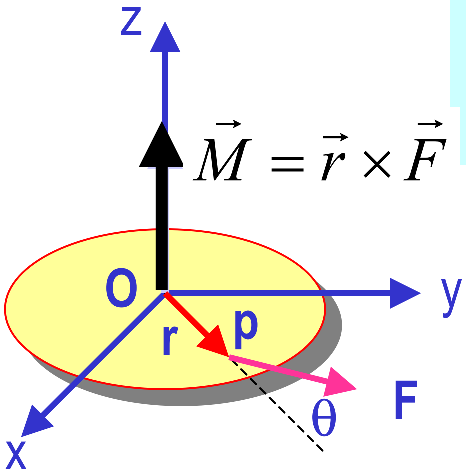
- Definicja: $$\boxed{\vec{M} = \vec{r} \times \vec{F}}$$

$$\left|\vec{M}\right| = r \cdot F \sin \phi = rF_{\phi}$$

- $$\underline{Gdy\ \vec{r} \parallel \vec{F} \quad to \quad \vec{M} = 0}$$



Moment siły



Na krążek zamocowany w punkcie O działa siła przyłożona w punkcie p.

Powoduje ona ruch obrotowy krążka wokół osi Z. w kierunku przeciwnym do ruchu wskazówek zegara.

Wartość momentu siły $M = Fr \sin \phi$

Moment siły jest wektorem prostopadłym do wektora $\vec{r}$ oraz wektora $\vec{F}$

$$\boxed{\overrightarrow{M} = \vec{r} \times \overrightarrow{F}}$$

Kierunek wektora $\vec{M}$ wyznacza reguła śruby prawoskrętnej.

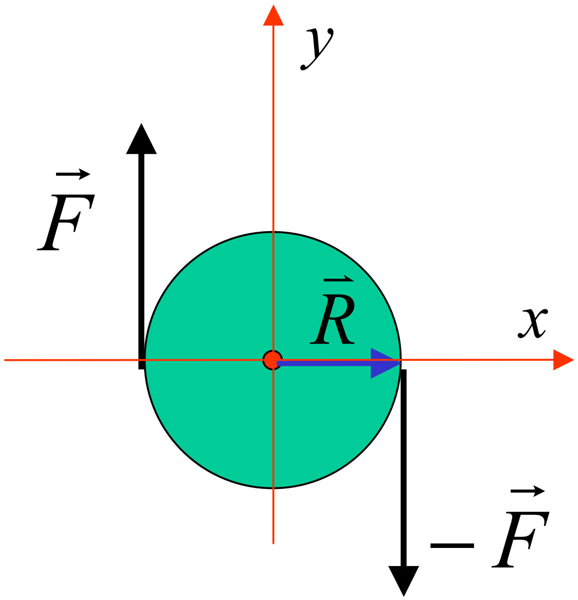
Iloczyn wektorowy dwu wektorów

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right| \sin \theta$$

Iloczyn skalarny dwu wektorów

$$C \equiv \vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos \theta$$

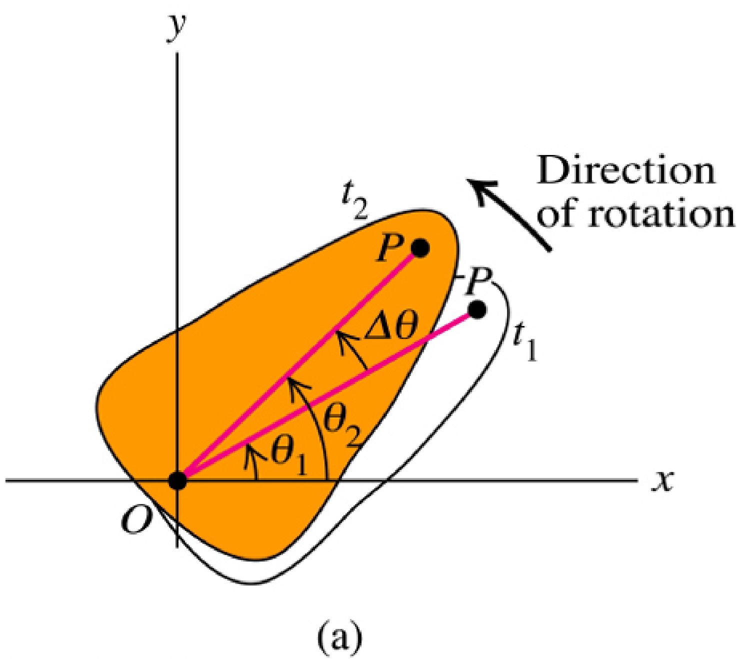


$$\sum \vec{F} = 0$$

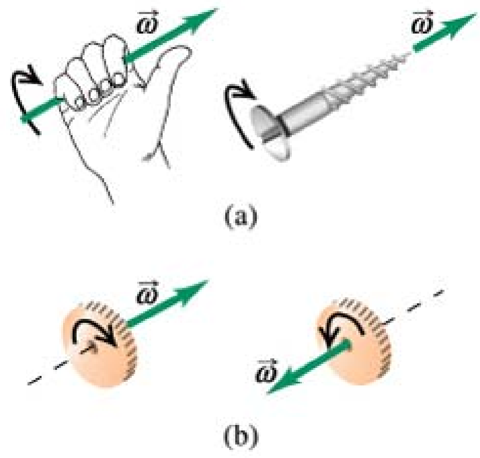
$$\vec{\varepsilon} \neq 0$$

$$\vec{M} = (-\vec{r} \times \vec{F}) + \vec{r} \times (-\vec{F})$$

$$\vec{\varepsilon} \propto \vec{M}$$



(a)

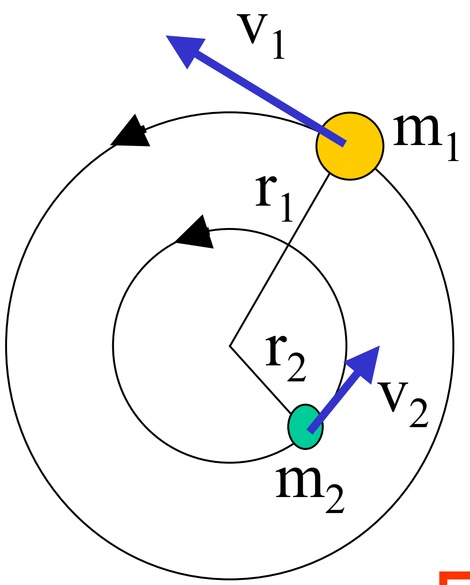


$$\omega = \frac{\Delta\theta}{\Delta t}$$

$$\varepsilon = \frac{\Delta\omega}{\Delta t}$$

Porównanie ruchów

Obrotowy	**Liniowy**
$\varepsilon = \text{const}$	$a = \text{const}$
$\omega = \omega_0 + \varepsilon t$	$v = v_0 + at$
$\alpha = \alpha_0 + \omega_0 t + \frac{1}{2}\varepsilon t^2$	$x = x_0 + v_0 t + \frac{1}{2}at^2$
$v = \omega R \qquad a = \varepsilon R$	



Ciała te poruszają się z różnymi wartościami prędkości liniowej, lecz z tą samą prędkością kątową ω.

$$v = \omega r$$

Energia kinetyczna układu

$$E_k = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2}\left(m_1 r_1^2 \omega^2 + m_2 r_2^2 \omega^2\right)$$

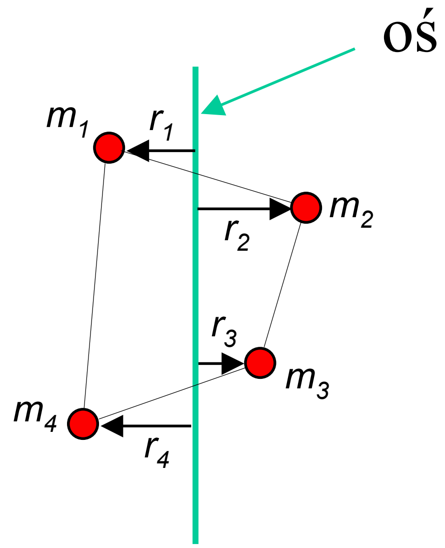
$$E_k = \frac{1}{2}\omega^2\left(m_1 r_1^2 + m_2 r_2^2\right) = \frac{1}{2} I \omega^2$$

Moment bezwładności $I = \left(m_1 r_1^2 + m_2 r_2^2\right)$

Moment bezwładności

- Dla układu punktów materialnych:

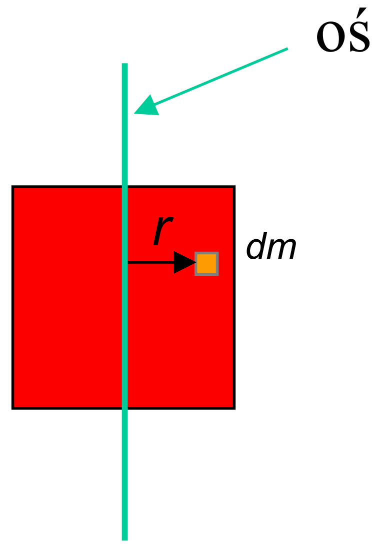
$$\boxed{I = \sum_{i=1}^{n} m_i r_i^2}$$



r_i - odległość od osi obrotu

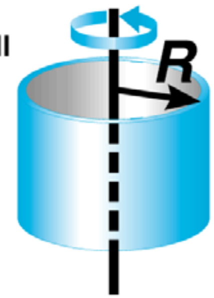
- Dla ciągłego rozkładu masy:

$$I = \int r^2 dm$$



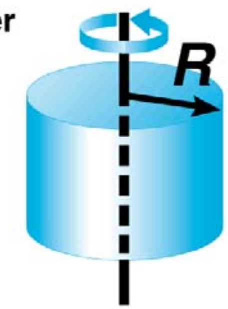
Hoop or cylindrical shell

$I = MR^2$



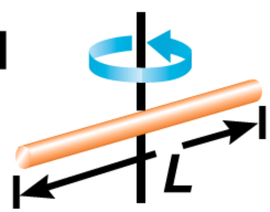
Solid cylinder or disk

$I = \frac{1}{2}MR^2$



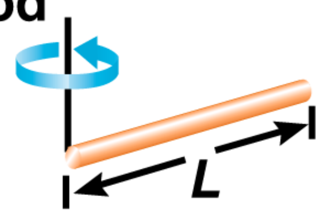
Long thin rod

$I = \frac{1}{12}ML^2$



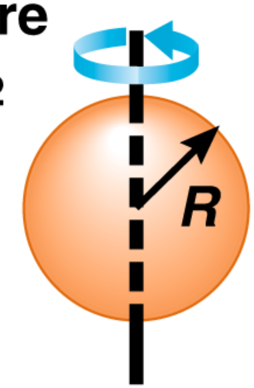
Long thin rod

$I = \frac{1}{3}ML^2$

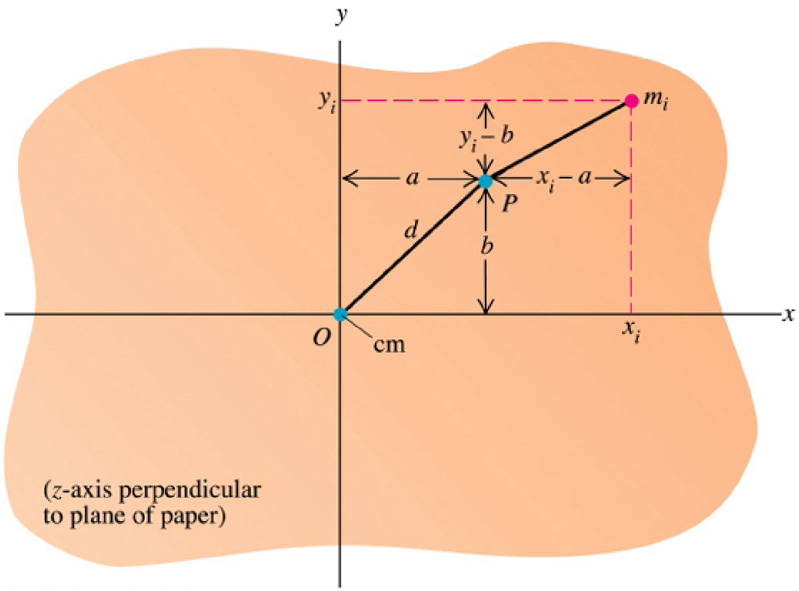


Solid sphere

$I = \frac{2}{5}MR^2$



Moment bezwładności względem



a) środka masy (punktu *cm*)

$$I_{cm} = \sum_i m_i (x_i^2 + y_i^2)$$

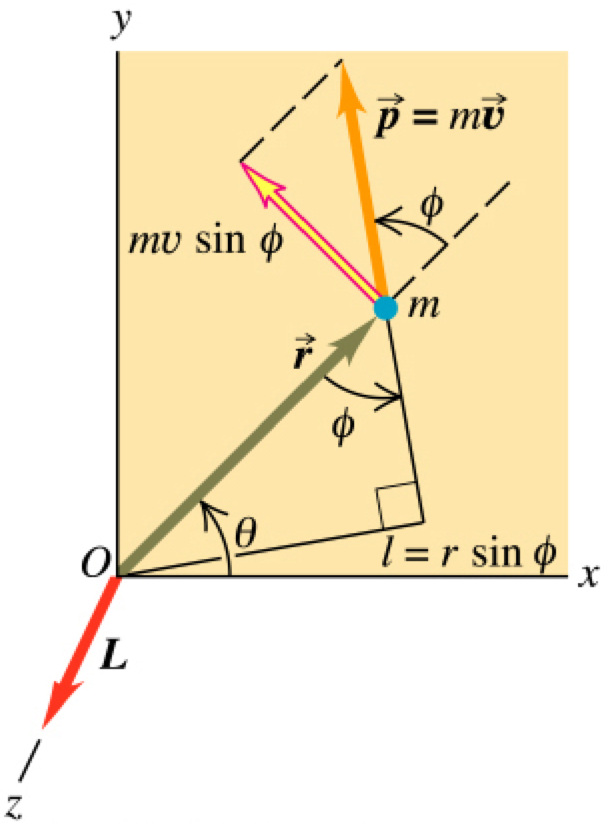
b) pewnego punktu P

$$I_P = \sum_i m_i ((x_i - a)^2 + (y_i - b)^2) =$$

$$\sum_i m_i (x_i^2 + y_i^2) - 2a \cancel{\sum_i m_i x_i} - 2b \cancel{\sum_i m_i y_i} + (a^2 + b^2) \sum_i m_i$$

$$\boxed{I_p = I_{cm} + M \cdot d^2}$$

Druga zasada dynamiki dla ruchu obrotowego



- Moment pędu $\vec{L}$

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

- Druga zasada dynamiki

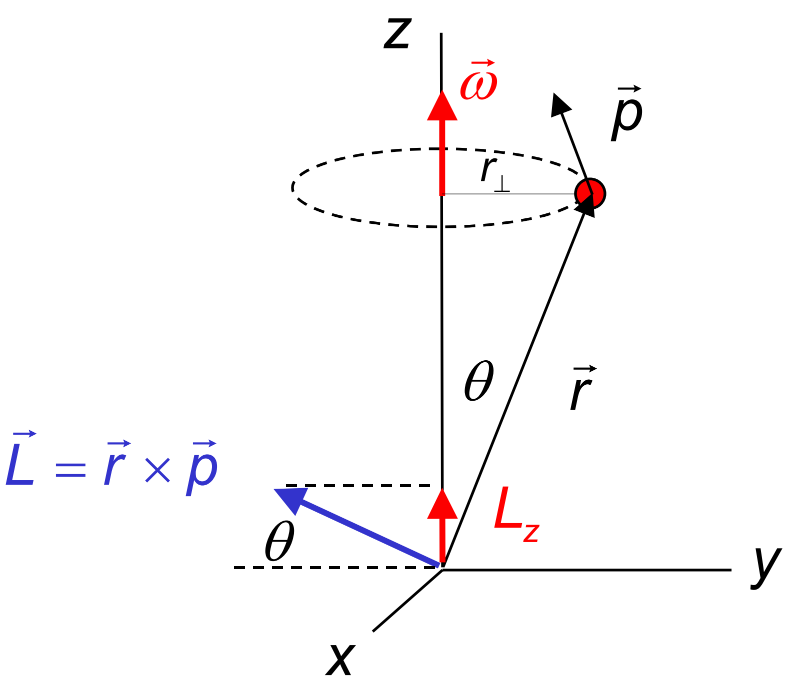
$$\Delta\vec{L} = \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p}$$

$$\frac{\Delta\vec{L}}{\Delta t} = \vec{r} \times \frac{\Delta\vec{p}}{\Delta t} + \underbrace{\frac{\Delta\vec{r}}{\Delta t} \times \vec{p}}_{= \vec{v} \times m\vec{v} = 0}$$

$\vec{v}$

$$\boxed{\frac{\Delta\vec{L}}{\Delta t} = \vec{r} \times \vec{F} \quad \Leftrightarrow \quad \frac{\Delta\vec{L}}{\Delta t} = \vec{M}}$$

Moment pędu



$$L_z = \left|\vec{L}\right| \sin\theta = pr \sin\theta = p r_\perp$$

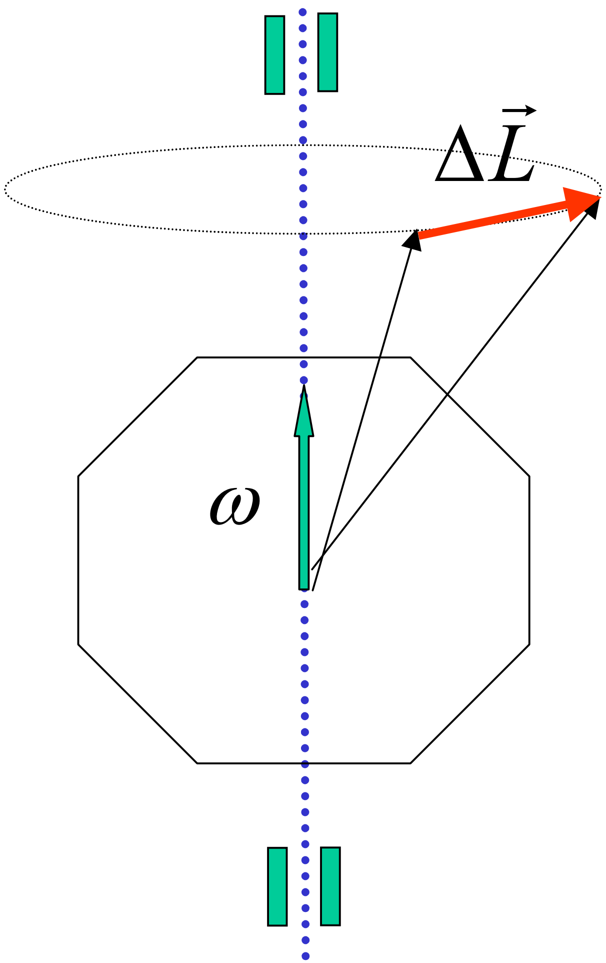
$$= m v r_\perp = m r_\perp^2 \omega = I\omega$$

$$\boxed{L_z = I\omega} \qquad \boxed{L_\perp \neq 0}$$

$\vec{L}$ *nie jest równolegy do* $\vec{\omega}$

$$\vec{L} = \vec{L}(t)$$

Ruch obrotowy względem nieruchomej osi



$$\vec{L} = \vec{L}(t)$$

$$\frac{\Delta\vec{L}}{\Delta t} = \vec{M} \quad \Rightarrow \quad \frac{\Delta\vec{L}_z}{\Delta t} = \vec{M}_z$$

$$L_z = I\omega$$

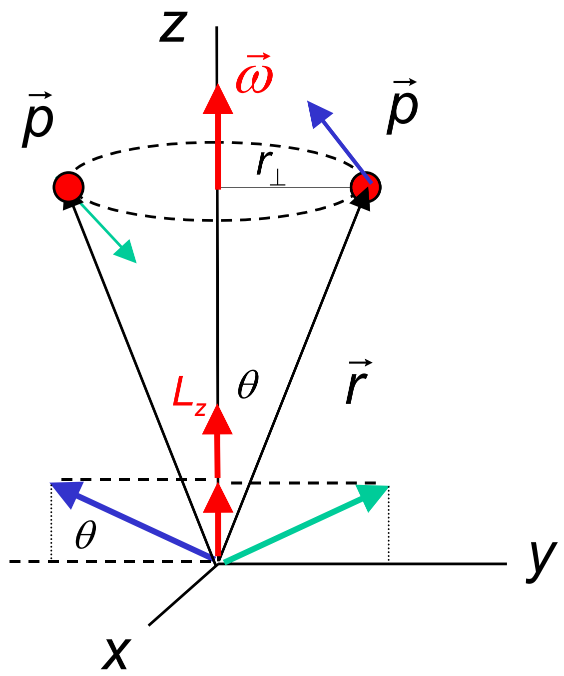
$$\frac{dL_z}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = I\varepsilon$$

$$\boxed{M_z = I\varepsilon}$$

Moment pędu

$$m_1 = m_2 = m$$

$$\vec{L} = \vec{r} \times \vec{p}$$



$$L_z = \left|\vec{L}\right|\sin\theta + \left|\vec{L}\right|\sin\theta = 2pr\sin\theta = 2pr_\perp$$

$$= 2mvr_\perp = 2mr_\perp^2\omega = I\omega$$

$$\boxed{L_z = I\omega} \qquad \boxed{L_\perp = 0}$$

Druga zasada dynamiki dla ruchu obrotowego

$$L = I\omega \quad \Rightarrow \quad \Delta L = I\Delta\omega + \Delta I\,\omega$$

$$\frac{\Delta L}{\Delta t} = I\underbrace{\frac{\Delta\omega}{\Delta t}}_{=\varepsilon} + \frac{\Delta I}{\Delta t}\omega$$

Jeżeli moment bezwładności nie ulega zmianie

$$\frac{\Delta L}{\Delta t} = I\varepsilon \quad \Rightarrow \quad \boxed{M = I\varepsilon}$$

Ruch postępowy	Ruch obrotowy
$\vec{F} = \dfrac{\Delta\vec{p}}{\Delta t}$; $\vec{F} = m\vec{a}$	$\dfrac{\Delta L}{\Delta t} = M;\quad M = I\varepsilon$

Prawo zachowania momentu pędu

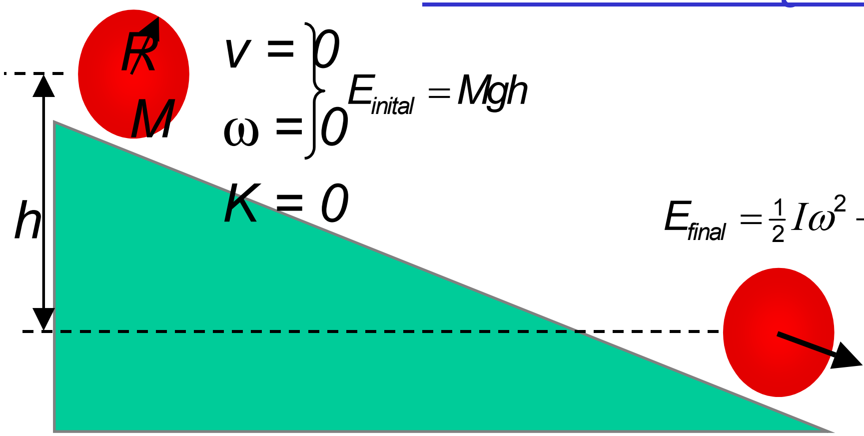
$$\boxed{\frac{\Delta \vec{L}}{\Delta t} = \vec{r} \times \vec{F} \quad \Leftrightarrow \quad \frac{\Delta \vec{L}}{\Delta t} = \vec{M}}$$

Jeżeli moment sił zewnętrznych jest równy zeru to moment pędu jest zachowany

$$\boxed{\vec{M} = 0 \quad \Leftrightarrow \quad \frac{\Delta \vec{L}}{\Delta t} = 0 \quad \Leftrightarrow \quad \Delta \vec{L} = 0}$$

Ruch postępowy	Ruch obrotowy
$\sum \vec{F} = 0 \Rightarrow \vec{p} = const$	$\vec{M} = 0 \Rightarrow \vec{L} = const$

Toczenie się ciał po równi pochyłej



$$E_{final} = \tfrac{1}{2} I\omega^2 + \tfrac{1}{2} Mv^2$$

$$v = \omega R$$

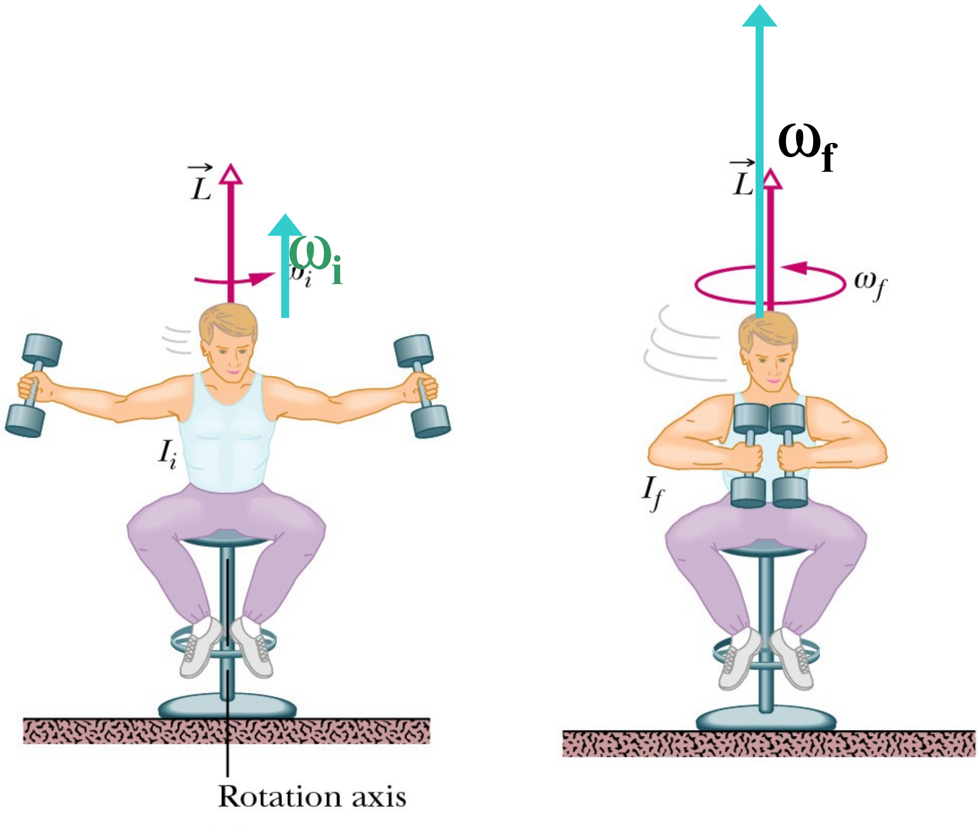
$v = \omega R$ oraz $I = cMR^2$

$$c = \begin{cases} 1 & \text{rura} \\ 1/2 & \text{walec} \\ 2/5 & \text{kula} \end{cases}$$

$$E_{final} = \tfrac{1}{2}\left(cMR^2\right)\left(\frac{v}{R}\right)^2 + \tfrac{1}{2}Mv^2 = \tfrac{1}{2}\left(c+1\right)Mv^2$$

$$E_{final} = E_{initial}$$

$$\tfrac{1}{2}\left(c+1\right)Mv^2 = Mgh \Rightarrow \boxed{v = \sqrt{2gh}\sqrt{\frac{1}{c+1}}}$$



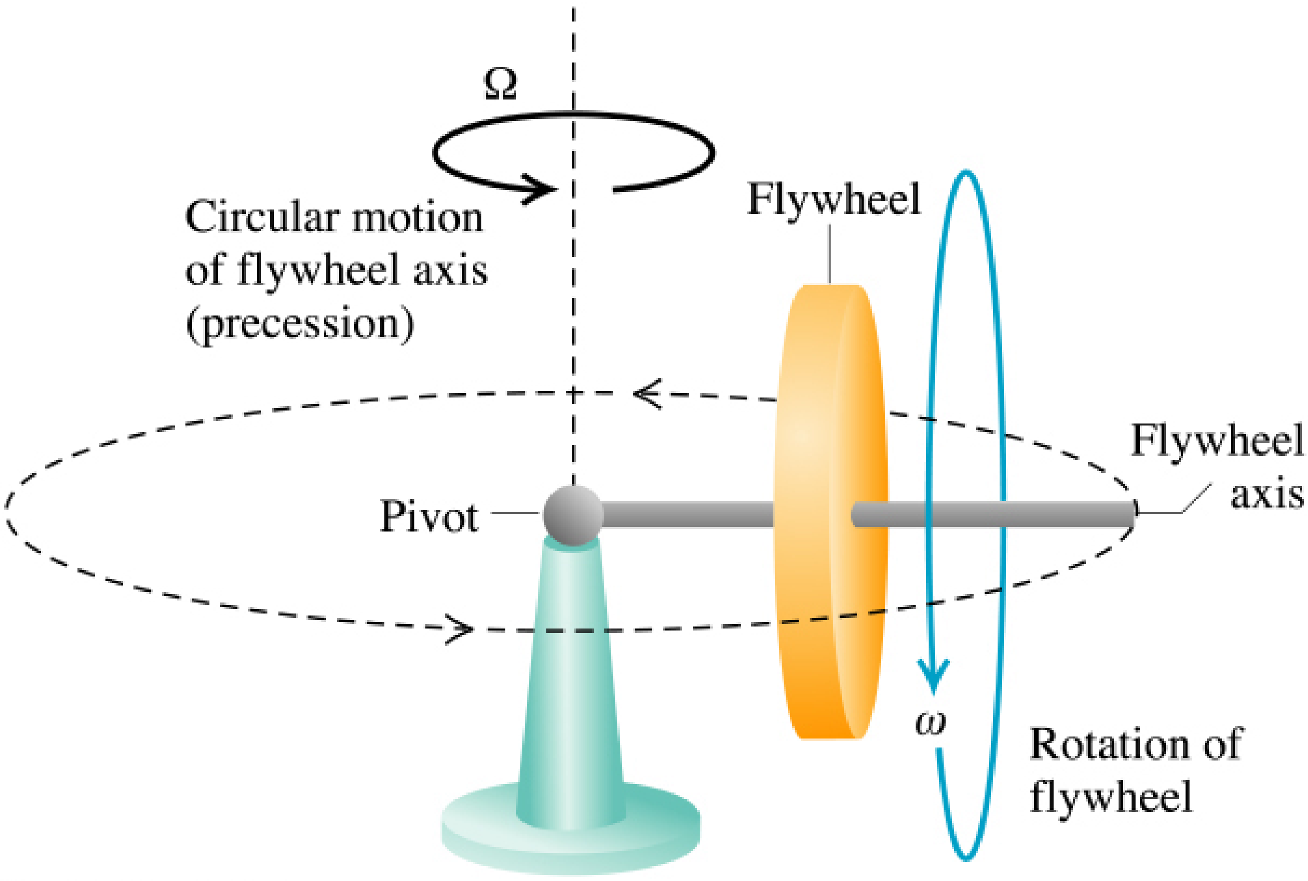
$$\vec{M}_{zew} = 0$$

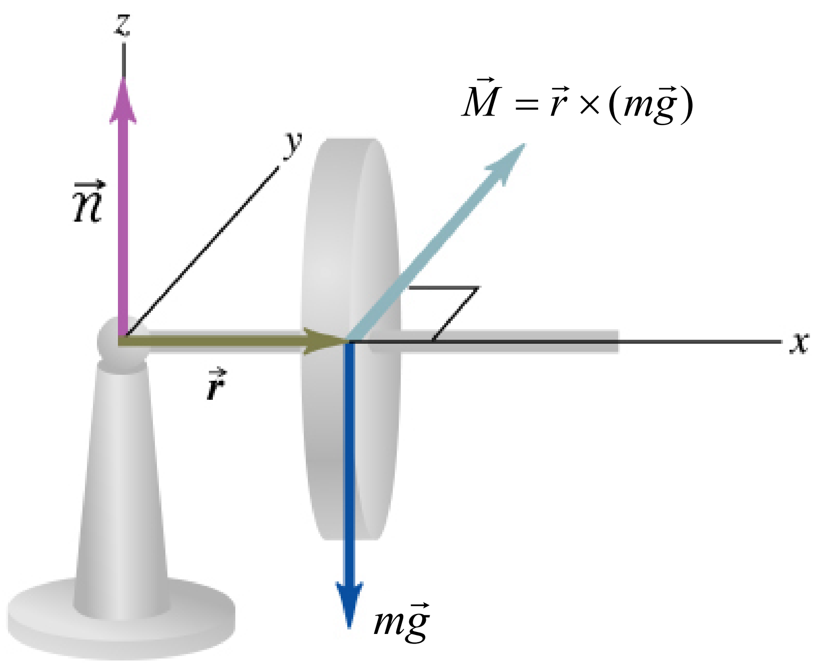
Moment pędu L jest zachowany

$$I_1\omega_1 = I_2\omega_2 \quad \Rightarrow \quad \omega_2 = \frac{I_1}{I_2}\omega_1$$

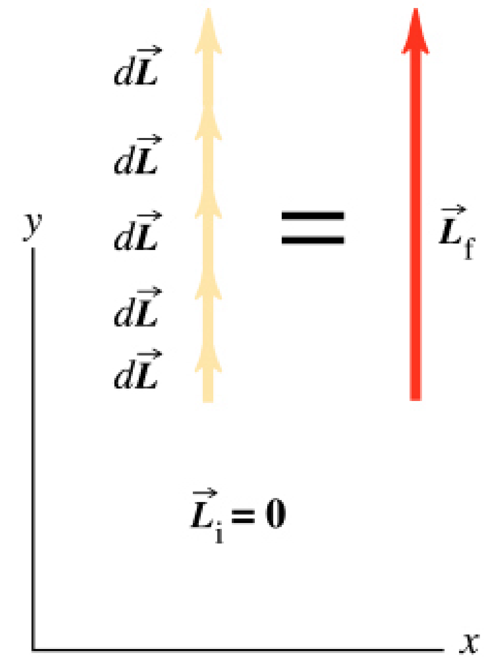
$$I_1 > I_2 \;\Rightarrow\; \omega_2 > \omega_1$$

$$\boxed{E_{k2} > E_{k1}}$$

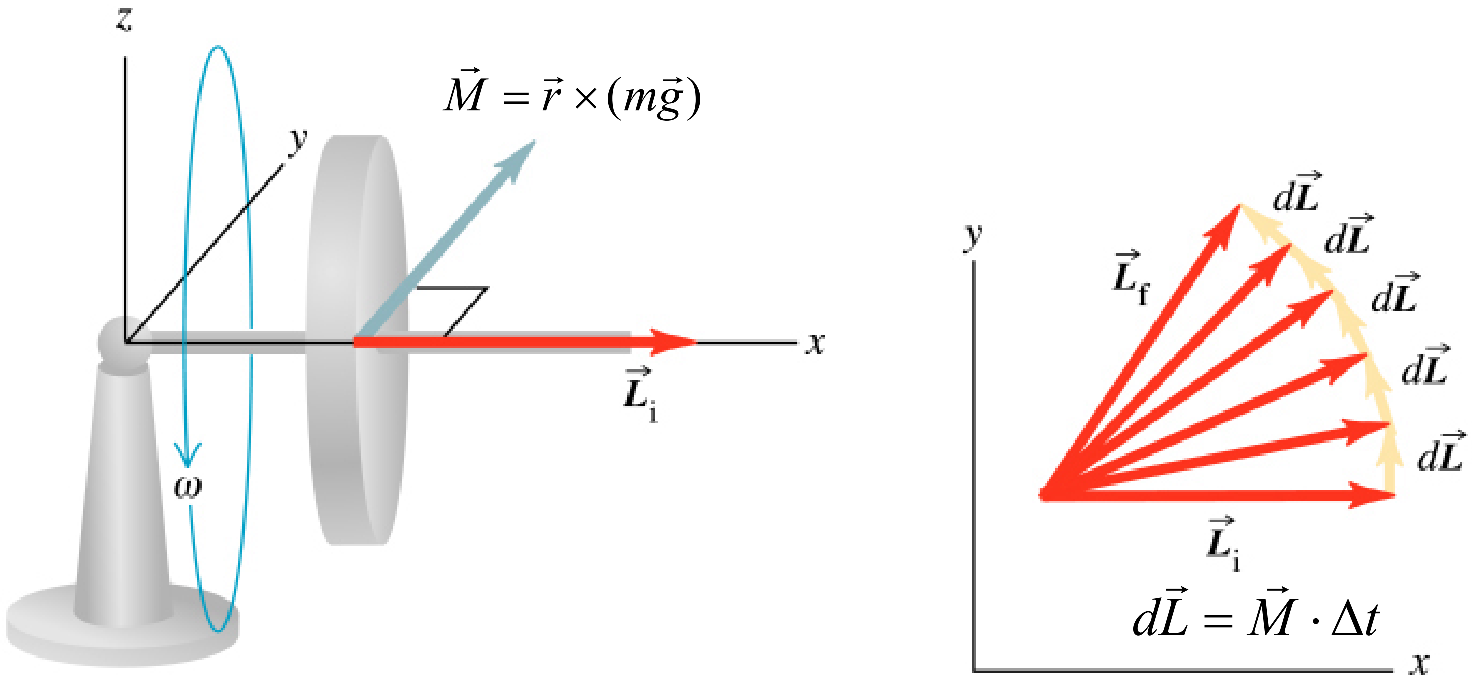

Ω
Circular motion
of flywheel axis
(precession)
Flywheel
Flywheel
axis
Pivot
ω
Rotation of
flywheel


z
y
$\vec{M} = \vec{r} \times (m\vec{g})$
$\vec{n}$
x
$\vec{r}$
$m\vec{g}$

(a)


$d\vec{L}$
$d\vec{L}$
y
$d\vec{L}$
=
$\vec{L}_f$
$d\vec{L}$
$d\vec{L}$
$\vec{L}_i = 0$
x

(b)



(a) (b)

$$\Omega = \frac{M}{L} = \frac{mg \cdot r}{I\omega}$$